

# Comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods

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## Abstract

The aim of this paper is to construct comodules of  $U_q(sl_2)$  and modules of  $SL_q(2)$  via quiver, where  $q$  is not a root of unity.

By embedding  $U_q(sl_2)$  into the path coalgebra  $k\mathcal{D}^c$ , where  $\mathbf{D}$  is the Gabriel quiver of  $U_q(sl_2)$  as a coalgebra, we obtain a basis of  $U_q(sl_2)$  in terms of combinations of paths of the quiver  $\mathbf{D}$ ; this special basis enables us to describe the category of  $U_q(sl_2)$ -comodules by certain representations of  $\mathbf{D}$ ; and this description further permits us to construct a class of modules of  $SL_q(2)$ , from certain representations of  $\mathbf{D}$ , via the duality between  $U_q(sl_2)$  and  $SL_q(2)$ .

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## 1. Introduction

Drinfeld [14] has established a duality, between the quantized enveloping algebra  $U_q(sl_2)$  and the quantum deformation  $SL_q(2)$  of the regular function ring on  $SL_2$  (see [17], VII). This has been extended between  $U_q(sl_n)$  and  $SL_q(n)$  by Takeuchi [30]. Therefore, any  $U_q(sl_n)$ -comodule (resp.  $SL_q(n)$ -comodule) can be endowed with an  $SL_q(n)$ -module structure (resp. a  $U_q(sl_n)$ -module), in a canonical way (see e.g. (5.1) below). However, this duality does not give  $U_q(sl_n)$ -comodules (resp.  $SL_q(n)$ -comodules) from  $SL_q(n)$ -modules (resp.  $U_q(sl_n)$ -modules).

Modules of  $U_q(\mathfrak{g})$  have been extensively studied (see e.g. [18,27,16]), and they depend on  $q$ : when  $q$  is not a root of unity, any finite-dimensional module is semi-simple, and the finite-dimensional simple module is a deformation of a finite-dimensional simple  $\mathfrak{g}$ -module. In [8] the prime and primitive spectra of (Lusztig's) quantized hyperalgebras (at roots of 1) are described. Another thing about  $U_q(\mathfrak{g})$  which depends on  $q$  is its coradical filtration [2,11,19,22]: when  $q$  is not a root of unity, the graded coalgebra  $U_q(\mathfrak{g})$  is coradically graded.

The study of  $SL_q(n)$ -comodules can also be found, e.g. in [24,7] (see also [15]). However there are few works on  $U_q(sl_n)$ -comodules. A possible reason for this lack might be that there are no proper tools to construct  $U_q(sl_n)$ -comodules. The aim of the present paper is to understand the  $U_q(sl_2)$ -comodules by using the quiver techniques.

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In the representation theory of algebras, quiver is a basic technique (see [1,26]). Recently, it has also shown powers in studying coalgebras and Hopf algebras. For example, one can construct path coalgebras of quivers, define the Gabriel quiver of a coalgebra, and embed a pointed coalgebra into the path coalgebra of its Gabriel quiver (see [9, 21,6]); after this embedding one can expect to study the comodules of the coalgebra by certain locally nilpotent representations of the quiver (see [4]); and this makes it possible to see the morphisms, the extensions, and even the Auslander–Reiten sequences (see e.g. [28]). One can also start from the Hopf quivers of groups to construct non-commutative, non-cocommutative pointed Hopf algebras (see [12]); this makes it possible to classify some Hopf algebras by quivers, whose bases can be explicitly given (see e.g. [5,23]).

Inspired by these ideas, in this paper, we construct comodules of  $U_q(sl_2)$  and modules of  $SL_q(2)$  via quiver methods, where  $q$  is not a root of unity. By embedding the quantized algebra  $U_q(sl_2)$  into the path coalgebra  $k\mathcal{D}^c$ , where  $\mathbf{D}$  is the Gabriel quiver of  $U_q(sl_2)$  as a coalgebra, we obtain a basis of  $U_q(sl_2)$  in terms of combinations of paths of the quiver  $\mathbf{D}$  (Theorem 3.5); this special basis enables us to describe the category of  $U_q(sl_2)$ -comodules by certain locally nilpotent representations of  $\mathbf{D}$  (Theorem 4.3); in particular, we can list all the indecomposable Schurian comodules of  $U_q(sl_2)$  (Theorem 4.7); and this description further permits us to construct a class of modules of the quantum special linear group  $SL_q(2)$ , from certain locally nilpotent representations of  $\mathbf{D}$ , via the duality between  $U_q(sl_2)$  and  $SL_q(2)$  (Theorem 5.2).

Note that these results also relate to the ones in [3] and [10], where the representations and prime ideals of  $SL_q(2)$  are studied.

## 2. Preliminaries

Throughout this paper, let  $k$  denote a field of characteristic zero, and  $q$  a non-zero element in  $k$  with  $q^2 \neq 1$ . For a  $k$ -space  $V$ , let  $V^*$  denote the dual space. Denote by  $\mathbb{Z}$  and  $\mathbb{N}_0$  the sets of integers and of non-negative integers, respectively.

### 2.1

By definition  $U_q(sl_2)$  is an associative  $k$ -algebra generated by  $E, F, K, K^{-1}$ , with relations (see e.g. [17], p. 122, or [16], p. 9)

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= q^2E, \quad KFK^{-1} = q^{-2}F, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

Then  $U_q(sl_2)$  has a Hopf structure with (see e.g. [17], p. 140)

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, & \Delta(F) &= K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \varepsilon(E) &= \varepsilon(F) = 0, & \varepsilon(K) &= \varepsilon(K^{-1}) = 1, \\ S(K) &= K^{-1}, & S(K^{-1}) &= K, & S(E) &= -EK^{-1}, & S(F) &= -KF. \end{aligned}$$

Note that  $U_q(sl_2)$  is a Noetherian algebra without zero divisors, and it has a basis  $\{K^l E^i F^j \mid i, j \geq 0, l \in \mathbb{Z}\}$  (see e.g. [17], p. 123).

By definition  $SL_q(2)$  is an associative  $k$ -algebra generated by  $a, b, c, d$ , with relations (see e.g. [17], p. 84)

$$\begin{aligned} ba &= qab, & db &= qbd, & ca &= qac, & dc &= qcd, & bc &= cb, \\ ad - da &= (q^{-1} - q)bc, & da - qbc &= 1. \end{aligned}$$

Then  $SL_q(2)$  has a Hopf structure with (see e.g. [17], p. 84)

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \end{aligned}$$

$$\begin{aligned}\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0, \\ S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a.\end{aligned}$$

## 2.2

By definition a duality between two Hopf algebras  $U$  and  $H$  is an algebra map  $\psi : H \longrightarrow U^*$ , such that  $\phi : U \longrightarrow H^*$  is also an algebra map and has the property

$$\psi(x)(S_U(u)) = \phi(u)(S_H(x))$$

for all  $u \in U$ ,  $x \in H$ , where  $\phi$  is defined by

$$\phi(u)(x) = \psi(x)(u),$$

and  $S_U$  and  $S_H$  are respectively the antipodes of  $U$  and  $H$ .

Suppose that there exists a duality between  $U$  and  $H$ . Then there also exists a duality between  $H$  and  $U$ ; and each  $U$ -comodule can be endowed with an  $H$ -module structure, and also each  $H$ -comodule can be endowed with a  $U$ -module.

We have the following well known duality between  $U_q(sl_2)$  and  $SL_q(2)$ . See Theorem VII.4.4 in [17].

**Lemma 2.3.** *There is a unique algebra map  $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$  such that*

$$\begin{aligned}\psi(a)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,0} q^l + \delta_{i,1} \delta_{j,1} q^l, & \psi(b)(K^l E^i F^j) &= \delta_{i,1} \delta_{j,0} q^l, \\ \psi(c)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,1} q^{-l}, & \psi(d)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,0} q^{-l},\end{aligned}$$

where  $\delta_{i,j}$  is the Kronecker symbol. This  $\psi$  is a duality between  $U_q(sl_2)$  and  $SL_q(2)$ .

Note that such a  $\psi$  is not injective. This duality was essentially introduced in [14], and has been extended to be a duality between  $U_q(sl_n)$  and  $SL_q(n)$  in [30].

## 2.4

A quiver  $Q = (Q_0, Q_1, s, t)$  is a datum, where  $Q$  is an oriented graph with  $Q_0$  the set of vertices and  $Q_1$  the set of arrows,  $s$  and  $t$  are two maps from  $Q_1$  to  $Q_0$ , such that  $s(a)$  and  $t(a)$  are respectively the starting vertex and terminating vertex of  $a \in Q_1$ . A path  $p$  of length  $l$  in  $Q$  is a sequence  $p = a_l \cdots a_2 a_1$  of arrows  $a_i$ ,  $1 \leq i \leq l$ , such that  $t(a_i) = s(a_{i+1})$  for  $1 \leq i \leq l-1$ . A vertex is regarded as a path of length 0. Denote by  $s(p)$  and  $t(p)$  the starting vertex and terminating vertex of  $p$ , respectively. Then  $s(p) = s(a_1)$  and  $t(p) = t(a_l)$ . If both  $Q_0$  and  $Q_1$  are finite sets, then  $Q$  is called a finite quiver. We will not restrict ourselves to finite quivers, but we assume the quivers considered are countable (i.e., both  $Q_0$  and  $Q_1$  are countable sets). For the quiver method to representations of algebras we refer to [1] and [26].

Given a quiver  $Q$ , we define the path coalgebra  $kQ^c$  (see [9]) as follows: the underlying space has as basis the set of all paths in  $Q$ , and the coalgebra structure is given by

$$\Delta(p) = \sum_{\beta\alpha=p} \beta \otimes \alpha$$

and

$$\varepsilon(p) = 0 \quad \text{if } l \geq 1, \quad \text{and} \quad \varepsilon(p) = 1 \quad \text{if } l = 0$$

for each path  $p$  of length  $l$ .

## 2.5

By a graded coalgebra we mean a coalgebra  $C$  with decomposition  $C = \bigoplus_{n \geq 0} C(n)$  of  $k$ -spaces such that

$$\Delta(C(n)) \subseteq \sum_{i+j=n} C(i) \otimes C(j), \quad \varepsilon(C(n)) = 0, \quad \forall n \geq 1.$$

Let  $C$  be a coalgebra. Following [29], the wedge of two subspaces  $V$  and  $W$  of  $C$  is defined to be the subspace

$$V \wedge W := \{c \in C \mid \Delta(c) \in V \otimes C + C \otimes W\}.$$

Let  $C_0$  be the coradical of  $C$ , i.e.,  $C_0$  is the sum of all simple subcoalgebras of  $C$ . Define  $C_n := C_0 \wedge C_{n-1}$  for  $n \geq 1$ . Then  $\{C_n\}_{n \geq 0}$  is called the coradical filtration of  $C$ .

Recall that a graded coalgebra  $C = \bigoplus_{n \geq 0} C(n)$  is said to be coradically graded, provided that  $\{C_n := \bigoplus_{i \leq n} C(i)\}_{n \geq 0}$  is exactly the coradical filtration of  $C$ . It was proved in [11], 2.2, that a graded coalgebra  $C = \bigoplus_{n \geq 0} C(n)$  is coradically graded if and only if  $C_0 = C(0)$  and  $C_1 = C(0) \oplus C(1)$ .

## 2.6

Let  $M$  be a  $C$ – $C$ -bicomodule over a coalgebra  $C$ . Denote by  $\text{Cot}_C(M)$  the corresponding cotensor coalgebra (see [13] for the definition and basic properties). This is a graded coalgebra with zeroth component  $C$  and first component  $M$ . By Proposition 11.1.1 in [29], the coradical of  $\text{Cot}_C(M)$  is contained in  $C$ . It follows that  $\text{Cot}_C(M)$  is coradically graded if and only if  $C$  is cosemisimple.

Note that a path coalgebra  $kQ^c$  is graded with the length grading, and it is coradically graded, and  $kQ^c \simeq \text{Cot}_{kQ_0}(kQ_1)$  (see [9], or [12]).

We need the following observation.

**Proposition 2.7.** *Let  $C = \bigoplus_{n \geq 0} C(n)$  be a graded coalgebra. Then*

- (i) *There is a unique graded coalgebra map  $\theta : C \rightarrow \text{Cot}_{C(0)}(C(1))$  such that  $\theta|_{C(i)} = \text{Id}$  for  $i = 0, 1$ .*
- (ii)  *$\theta(x) = \pi^{\otimes n+1} \circ \Delta^n(x)$  for all  $x \in C(n+1)$  and  $n \geq 1$ , where  $\pi : C \rightarrow C(1)$  is the projection, and  $\Delta^n = (\text{Id} \otimes \Delta^{n-1}) \circ \Delta$  for all  $n \geq 1$ , with  $\Delta^0 = \text{Id}$ .*
- (iii) *If  $C$  is coradically graded, then  $\theta$  is injective.*
- (iv) *If  $C(0)$  is cosemisimple, and  $\theta$  is injective, then  $C$  is coradically graded.*

**Proof.** Clearly,  $C(0)$  is a subcoalgebra and  $C(1)$  is naturally a  $C(0)$ – $C(0)$ -bicomodule, and hence we have the corresponding cotensor coalgebra  $\text{Cot}_{C(0)}(C(1))$ . The statements (i) and (ii) follow from the universal property of a cotensor coalgebra (see e.g. [25], or [12]).

For statement (iii), if  $C$  is coradically graded, then  $C_1 = C(0) \oplus C(1)$ . It follows that  $\theta|_{C_1}$  is injective, and hence  $\theta$  is injective, by a theorem due to Heynemann and Radford (see e.g. [20], 5.3.1).

If  $C(0)$  is cosemisimple, then  $\text{Cot}_{C(0)}(C(1))$  is coradically graded. The injectivity of  $\theta$  implies that  $C$  is a graded subcoalgebra of  $\text{Cot}_{C(0)}(C(1))$ . Thus  $C$  is also coradically graded. ■

## 2.8

Consider a special case of Proposition 2.7 where  $C(0)$  is a group-like coalgebra (i.e., it has a basis consisting of group-like elements; or equivalently,  $C(0)$  is cosemisimple and pointed). In this case we have  $C(0) = kG(C)$ , where

$$G(C) := \{g \in C \mid \Delta(g) = g \otimes g, \varepsilon(g) = 1\}.$$

Since  $C(1)$  is a  $C(0)$ – $C(0)$ -bicomodule, it follows that

$$C(1) = \bigoplus_{g, h \in G} {}^h C(1)^g,$$

where  ${}^h C(1)^g = \{c \in C(1) \mid \Delta(c) = c \otimes g + h \otimes c\}$ . Define a quiver  $Q = Q(C)$  as follows: the set of vertices is  $G$ , and there are exactly  $t_{gh}$  arrows from vertex  $g$  to vertex  $h$ , where  $t_{gh} = \dim_k {}^h C(1)^g$ . Then by the universal property of a cotensor coalgebra (and hence of a path coalgebra), there is a coalgebra isomorphism  $\text{Cot}_{C(0)}(C(1)) \simeq kQ^c$ , by identifying the elements of  $G(C)$  with the vertices of  $Q$  and a basis of  ${}^h C(1)^g$  with the arrows from  $g$  to  $h$ .

Note that the quiver  $Q(C)$  is in general not the Gabriel quiver of  $C$ . If the graded coalgebra  $C = \bigoplus_{n \geq 0} C(n)$  is coradically graded, then  $Q(C)$  is exactly the Gabriel quiver of  $C$ . For the equivalent definitions of the Gabriel quiver of a coalgebra we refer to [6], Section 2 (see also [9,21,28]). By Proposition 2.7 we have

**Corollary 2.9.** Assume that  $C = \bigoplus_{n \geq 0} C(n)$  is a graded coalgebra with  $C(0)$  group-like. Let  $Q(C)$  be the quiver associated with  $C$  defined as above. Then

- (i) There is a graded coalgebra map  $\theta : C \longrightarrow kQ(C)^c$ .
- (ii)  $\theta$  is injective if and only if  $C$  is coradically graded. In this case,  $Q(C)$  is exactly the Gabriel quiver of  $C$ .

### 3. $U_q(sl_2)$ as a subcoalgebra of a path coalgebra

In this section, we embed  $U_q(sl_2)$  into the path coalgebra of the Gabriel quiver  $\mathbf{D}$  of  $U_q(sl_2)$ , and then give a new basis of  $U_q(sl_2)$  in terms of combinations of paths in  $\mathbf{D}$ , where  $q$  is not a root of unity.

Although bases of  $U_q(sl_2)$  are already available, this new basis of  $U_q(sl_2)$  given here, which is in terms of combinations of paths in  $\mathbf{D}$ , will enable us to describe the category of  $U_q(sl_2)$ -comodules, in terms of  $k$ -representations of the quiver  $\mathbf{D}$ .

#### 3.1

For each non-negative integer  $n$ , let  $C(n)$  be the subspace of  $U_q(sl_2)$  with basis the set  $\{K^l E^i F^j \mid i, j \in \mathbb{N}_0, i+j = n, l \in \mathbb{Z}\}$ . Then

$$U_q(sl_2) = \bigoplus_{n \geq 0} C(n)$$

is a graded coalgebra (see for example Proposition VII.1.3 in [17]) with

$$G(U_q(sl_2)) = \{K^l \mid l \in \mathbb{Z}\}, \quad \text{and} \quad C(0) = \bigoplus_{l \in \mathbb{Z}} kK^l.$$

We have in  $C(1)$

$$\begin{aligned} \Delta(K^{l-1}E) &= K^{l-1} \otimes K^{l-1}E + K^{l-1}E \otimes K^l, \\ \Delta(K^lF) &= K^{l-1} \otimes K^lF + K^lF \otimes K^l. \end{aligned}$$

Note that  $C(1)$  has a basis  $\{K^lE, K^lF \mid l \in \mathbb{Z}\}$ ;

$$K^{l_2}C(1)^{K^{l_1}} = 0 \quad \text{for } (l_1, l_2) \neq (l, l-1), l \in \mathbb{Z},$$

and that for each  $l \in \mathbb{Z}$  we have

$$K^{l-1}C(1)^{K^l} = kK^{l-1}E \oplus kK^lF, \quad l \in \mathbb{Z}.$$

Therefore, the quiver of  $U_q(sl_2)$  as defined in 2.8 is of the form

$$\cdots \cdots \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots \cdots$$

We will denote this quiver by  $\mathbf{D}$  in this paper.

#### 3.2

We fix some notations. Index the vertices of  $\mathbf{D}$  by integers, i.e.,  $\mathcal{D}_0 = \{e_l \mid l \in \mathbb{Z}\}$ ; there are two arrows from  $e_l$  to  $e_{l-1}$  for each integer  $l$ . Put  $I = \{1, -1\}$  and let  $I^n$  be the Cartesian product (understand  $I^0 := \{0\}$ ). Define  $\mathcal{I} = \bigcup_{n \geq 0} I^n$ . For each  $v \in \mathcal{I}$ , define  $|v| = n$  if  $v \in I^n$ . Write  $v$  as  $v = (v_1, \dots, v_n)$ , where  $v_j = 1$  or  $-1$  for each  $j$ . For any integer  $l$  and  $v \in \mathcal{I}$ , define

$$P_l^{(v)} = a_{|v|} \cdots a_2 a_1$$

to be the concatenated path in  $\mathbf{D}$  starting at  $e_l$  of length  $|v|$ , where the arrow  $a_j$  is the upper arrow if  $v_j = 1$ , and the lower one otherwise,  $1 \leq j \leq |v|$ .

For example,  $P_l^{(0)}$  is understood to be the vertex  $e_l$ ;  $P_l^{(1)}$  (resp.  $P_l^{(-1)}$ ) is the upper (resp. lower) arrow starting at the vertex  $e_l$  in  $\mathbf{D}$ . Clearly,

$$\{P_l^{(v)} = P_{l-|v|+1}^{(v_{|v|})} \cdots P_{l-1}^{(v_2)} P_l^{(v_1)} \mid l \in \mathbb{Z}, v \in \mathcal{I}\}$$

is the set of all paths in  $\mathbf{D}$ .

As an application of Corollary 2.9 we have

**Lemma 3.3.** *There is a unique graded coalgebra map  $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$  such that  $\theta(K^l) = e_l$ ,  $\theta(K^{l-1}E) = P_l^{(1)}$ , and  $\theta(K^lF) = P_l^{(-1)}$ , for each integer  $l$ .*

*Moreover, if  $q$  is not a root of unity, then  $\theta$  is injective. In this case,  $\mathbf{D}$  is the Gabriel quiver of the coalgebra  $U_q(sl_2)$ .*

**Proof.** The existence of  $\theta$  follows directly from Corollary 2.9, and the uniqueness follows from the universal property of a path coalgebra. Note that if  $q$  is not a root of unity, then the graded coalgebra  $U_q(sl_2) = \bigoplus_{n \geq 0} C(n)$  is coradically graded (see [11]). ■

### 3.4

For  $v \in I^n \subset \mathcal{I}$ , put

$$T_v := \{t \mid 1 \leq t \leq n, v_t = 1\}, \quad \chi(v) := q^{2 \sum_{t \in T_v} t}, \quad \text{if } n \geq 1, T_v \neq \emptyset;$$

$$\chi(v) := 1, \quad \text{otherwise.}$$

For each  $l \in \mathbb{Z}, n \in \mathbb{N}_0, 0 \leq i \leq n$ , set

$$b(l, n, i) := \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \in k\mathcal{D}^c.$$

For example, we have

$$b(l, 0, 0) = e_l, \quad b(l, 1, 0) = P_l^{(-1)}, \quad b(l, 1, 1) = q^2 P_l^{(1)},$$

$$b(l, 2, 0) = P_l^{(-1, -1)}, \quad b(l, 2, 2) = q^6 P_l^{(1, 1)},$$

$$b(l, 2, 1) = q^2 P_l^{(1, -1)} + q^4 P_l^{(-1, 1)}.$$

The main theorem of this section is

**Theorem 3.5.** *Assume that  $q$  is not a root of unity. Then as a coalgebra  $U_q(sl_2)$  is isomorphic to the subcoalgebra of  $k\mathcal{D}^c$  with the set of basis*

$$\{b(l, n, i) \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\}.$$

For a non-zero element  $q$  in  $k$ , and non-negative integers  $n \geq m$ , the Gaussian binomial coefficient is defined to be

$$\binom{n}{m}_q = \frac{n!_q}{m!_q (n-m)!_q}$$

where  $n!_q := 1_q 2_q \cdots n_q$ ,  $0!_q := 1$ ,  $n_q := 1 + q + \cdots + q^{n-1}$ .

Given a positive integer  $n$ , and two vectors  $s = (s_0, s_1, \dots, s_{n-1}), r = (r_0, r_1, \dots, r_{n-1}) \in \mathbb{N}_0^n$  with the property

$$s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \quad r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

set

$$c(s, r) := \binom{s_0}{s_1}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2} \binom{r_0}{r_1}_{q^{-2}} \cdots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}} q^{2 \sum_{t=1}^{n-1} r_t (s_{t-1} - s_t)}.$$

**Lemma 3.6.** Put  $E' := K^{-1}E \in U_q(sl_2)$ . Then for any non-negative integers  $i$  and  $j$ , with  $n := i + j \geq 1$ , we have

$$\Delta^{n-1}(K^l E'^i F^j) = \sum_{s,r} c(s, r) K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1} \otimes \dots \\ \otimes K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes K^l E'^{s_{n-1}} F^{r_{n-1}}$$

where the sum runs over all the  $r$  and  $s$  with  $s_0 = i$  and  $r_0 = j$ .

**Proof.** It suffices to prove the formula for  $n \geq 2$ . Note that

$$\Delta(E'^i) = \Delta(E')^i = (K^{-1} \otimes E' + E' \otimes 1)^i \\ = \sum_{s_1=0}^i \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1}.$$

So

$$\Delta^{n-1}(E'^i) = (Id \otimes \Delta^{n-2}) \left( \sum_{s_1=0}^i \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1} \right) \\ = \sum_{s_1=0}^i \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes \Delta^{n-2}(E'^{s_1}).$$

By induction we have

$$\Delta^{n-1}(E'^i) = \sum_{0 \leq s_{n-1} \leq s_{n-2} \leq \dots \leq s_1 \leq i} \binom{i}{s_1}_{q^2} \binom{s_1}{s_2}_{q^2} \dots \binom{s_{n-2}}{s_{n-1}}_{q^2} \\ \times K^{-s_1} E'^{i-s_1} \otimes K^{-s_2} E'^{s_1-s_2} \otimes \dots \otimes K^{-s_{n-1}} E'^{s_{n-2}-s_{n-1}} \otimes E'^{s_{n-1}}.$$

Similarly, we have

$$\Delta^{n-1}(F^j) = \sum_{0 \leq r_{n-1} \leq r_{n-2} \leq \dots \leq r_1 \leq j} \binom{j}{r_1}_{q^{-2}} \binom{r_1}{r_2}_{q^{-2}} \dots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}} \\ \times K^{-r_1} F^{j-r_1} \otimes K^{-r_2} F^{r_1-r_2} \otimes \dots \otimes K^{-r_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes F^{r_{n-1}}.$$

Now the formula follows from  $\Delta^{n-1}(K^l E'^i F^j) = \Delta^{n-1}(K^l) \Delta^{n-1}(E'^i) \Delta^{n-1}(F^j)$  and the identity

$$E'^m K^{-t} = q^{2mt} K^{-t} E'^m, \quad m, t \in \mathbb{N}_0. \quad \blacksquare$$

### 3.7. Proof of Theorem 3.5

Since  $q$  is not a root of unity, it follows from Lemma 3.3 that there is a coalgebra embedding  $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$ . Put  $E' := K^{-1}E$ . Then  $\{K^l E'^i F^j \mid i, j \in \mathbb{N}_0, l \in \mathbb{Z}\}$  is a basis of  $U_q(sl_2)$ . Note that

$$\theta(K^l) = e_l, \quad \theta(K^l E') = P_l^{(1)}, \quad \theta(K^l F) = P_l^{(-1)}.$$

Denote by  $\pi$  the projection  $U_q(sl_2) \longrightarrow C(1) \simeq k\mathcal{D}_1$ . Then

$$\pi(K^{l-1} E) = P_l^{(1)}, \quad \pi(K^l F) = P_l^{(-1)}, \quad \pi(K^l E'^i F^j) = 0 \quad \text{for } i + j \geq 2.$$

By Proposition 2.7(ii) we have

$$\theta(K^l E'^i F^j) = \pi^{\otimes n} \circ \Delta^{n-1}(K^l E'^i F^j)$$

where  $n = i + j$ , and both  $i$  and  $j$  are positive integers. By Lemma 3.6 and the definition of  $\pi$  we have

$$\theta(K^l E'^i F^j) = \sum_{s,r} c(s, r) \pi(K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1}) \dots \\ \cdot \pi(K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}}) \cdot \pi(K^l E'^{s_{n-1}} F^{r_{n-1}})$$

where the dot means the concatenation of paths, and the sum runs over all the vectors  $s = (s_0, s_1, \dots, s_{n-1})$ ,  $r = (r_0, r_1, \dots, r_{n-1}) \in \mathbb{N}_0^n$ , with

$$i = s_0 \geq s_1 \geq \dots \geq s_{n-1}, \quad j = r_0 \geq r_1 \geq \dots \geq r_{n-1},$$

such that for each  $t$ ,  $1 \leq t \leq n$ , either

$$s_{t-1} - s_t = 1, \quad r_{t-1} - r_t = 0,$$

or

$$s_{t-1} - s_t = 0, \quad r_{t-1} - r_t = 1,$$

where  $s_n$  and  $r_n$  are understood to be zero.

Now, for such a pair  $(s, r)$ , define  $v = (v_1, \dots, v_n) \in I^n$  as follows:

$$v_{n-t+1} = 1, \quad \text{if } s_{t-1} - s_t = 1, r_{t-1} - r_t = 0;$$

and

$$v_{n-t+1} = -1, \quad \text{if } s_{t-1} - s_t = 0, r_{t-1} - r_t = 1,$$

for  $1 \leq t \leq n$ . Write  $(s, r)$  as  $(s, r) = (s_v, r_v)$ .

Since  $(s_{t-1} + r_{t-1}) - (s_t + r_t) = 1$  and  $s_n + r_n = 0$ , it follows that  $s_t + r_t = n - t$  for  $1 \leq t \leq n - 1$ . Therefore, we have

$$\begin{aligned} \theta(K^l E^i F^j) &= \sum_{s_v, r_v} c(s_v, r_v) P_l^{(v)} \\ &= \sum_{v \in I^n, |T_v|=i} c(s_v, r_v) P_l^{(v)}. \end{aligned}$$

Note that for  $s_v = (i = s_0, s_1, \dots, s_{n-1})$ , any number in the sequence  $i - s_1, \dots, s_{n-2} - s_{n-1}, s_{n-1}$  is either 1 or 0, and that the number of 1s in the sequence is exactly  $i$ . This implies

$$\binom{i}{s_1}_{q^2} \dots \binom{s_{n-2}}{s_{n-1}}_{q^2} = i!_{q^2}.$$

In order to compute  $c(s_v, r_v)$ , let  $T_v = \{t_1, \dots, t_i\}$ , with  $1 \leq t_1 < \dots < t_i \leq n$ . By an analysis on the components of

$$r_v = (j = r_0, \dots, r_{n-t_i}, r_{n-t_i+1}, \dots, r_{n-t_{(i-1)}}, \dots, r_{t_1}, \dots, r_{n-1}),$$

we observe that  $r_{n-t_i} = r_{n-t_i+1}$  since  $v_{t_i} = 1$ , and  $j = r_0, \dots, r_{n-t_i}$  are pairwise different. It follows that

$$r_{n-t_i} = j - n + t_i.$$

A similar analysis shows that

$$r_{n-t_x} = j - n + t_x + (i - x), \quad x = 1, \dots, i.$$

It follows that

$$\begin{aligned} \sum_{t=1}^{n-1} r_t (s_{t-1} - s_t) &= \sum_{1 \leq t \leq n-1, v_{n-t+1}=1} r_t \\ &= r_{n-t_1} + \dots + r_{n-t_i} \\ &= (t_1 + \dots + t_i) - \frac{i(i+1)}{2}. \end{aligned}$$

This shows

$$c(s_v, r_v) = i!_{q^2} j!_{q-2} q^{-i(i+1)} \chi(v),$$



and hence

$$\begin{aligned}\theta(K^l E^i F^j) &= i!_{q^2} j!_{q^{-2}} q^{-i(i+1)} \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \\ &= i!_{q^2} j!_{q^{-2}} q^{-i(i+1)} b(l, n, i)\end{aligned}\quad (3.1)$$

for  $n = i + j \geq 2$  and any integer  $l$ . Thus  $U_q(sl_2) \simeq \theta(U_q(sl_2))$  is spanned by

$$\{b(l, n, i) \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\},$$

while this set is obviously  $k$ -linearly independent. This completes the proof. ■

#### 4. Comodules of $U_q(sl_2)$

In this section, by applying [Theorem 3.5](#) we will characterize the category of the  $U_q(sl_2)$ -comodules in terms of the representations of the quiver **D** (see [Theorem 4.3](#)), and then list all the indecomposable Schurian  $U_q(sl_2)$ -comodules (see [Theorem 4.7](#)), where  $q$  is not a root of unity.

##### 4.1

Let  $Q$  be a quiver (not necessarily finite). By definition a  $k$ -representation of  $Q$  is a datum  $V = (V_e, f_a; e \in Q_0, a \in Q_1)$ , where  $V_e$  is a  $k$ -space for each  $e \in Q_0$ , and  $f_a : V_{s(a)} \rightarrow V_{t(a)}$  is a  $k$ -linear map for each  $a \in Q_1$ . Set  $f_p := f_{a_l} \circ \cdots \circ f_{a_1}$  for each path  $p = a_l \cdots a_1$ , where each  $a_i$  is an arrow,  $1 \leq i \leq l$ . Set  $f_e := Id$  for  $e \in Q_0$ . Then  $f_p$  is a  $k$ -linear map from  $V_{s(p)}$  to  $V_{t(p)}$ . A morphism  $\phi : (V_e, f_a; e \in Q_0, a \in Q_1) \rightarrow (W_e, g_a; e \in Q_0, a \in Q_1)$  is a datum  $\phi = (\phi_e; e \in Q_0)$  such that

$$\phi_{t(a)} f_a = g_a \phi_{s(a)}$$

for each  $a \in Q_1$ . Denote by  $\text{Rep}(k, Q)$  the category of the  $k$ -representations of  $Q$ . We refer the representations of quivers to [1] and [26].

A representation  $V = (V_e, f_a; e \in Q_0, a \in Q_1)$  is said to be locally nilpotent, provided that for each  $e \in Q_0$  and each  $m \in V_e$ , there are only finitely many paths  $p$  starting at  $e$  such that  $f_p(m) \neq 0$ .

It was observed by Chin and Quinn that there is an equivalence between the category of the right  $kQ^c$ -modules and the category of the locally nilpotent representations of  $Q$  (see [4]). The functors can be seen from the following.

For a right  $kQ^c$ -comodule  $(M, \rho)$ , define for each  $e \in Q_0$

$$M_e := \{m \in M \mid \rho_0(m) = m \otimes e\}$$

where  $\rho_0 = (Id \otimes \pi_0)\rho$ , and  $\pi_0 : kQ^c \rightarrow kQ_0$  is the projection. For every path  $p$  there is a unique  $k$ -linear map  $f_p : M_{s(p)} \rightarrow M_{t(p)}$ , such that for each  $m \in M_{s(p)}$  there holds

$$\rho(m) = \sum_{s(p')=s(p)} f_{p'}(m) \otimes p'$$

where  $p'$  runs over all the paths with  $s(p') = s(p)$ . In this way we obtain a  $k$ -representation  $(M_e, f_a; e \in Q_0, a \in Q_1)$  of  $Q$  satisfying  $f_p = f_\beta f_\alpha$  for any path  $p = \beta\alpha$ . By construction it is clearly a locally nilpotent representation. Note that  $M$  is a  $kQ_0$ -comodule with  $\rho_0$ . Since  $kQ_0$  is group-like, it follows that we have a  $kQ_0$ -comodule decomposition

$$M = \bigoplus_{e \in Q_0} M_e. \quad (4.1)$$

Conversely, given a locally nilpotent representation  $V = (V_e, f_a; e \in Q_0, \alpha \in Q_1)$  of  $Q$ , define

$$M := \bigoplus_{e \in Q_0} V_e$$

and  $\rho : M \rightarrow M \otimes kQ^c$  by

$$\rho(m) := \sum_{s(p)=e} f_p(m) \otimes p$$

for each  $m \in V_e$  (where  $f_e$  is understood to be  $Id$  for  $e \in Q_0$ ). Then  $\rho$  is well defined since  $V$  is locally nilpotent and  $(M, \rho)$  is a right  $kQ^c$ -comodule.

#### 4.2

Keep the notations in 3.2. Given a representation  $V = (V_l, V_a; e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$  of the quiver  $\mathbf{D}$ , define  $f_l^{(v)} := f_{P_l^{(v)}}$ , for each integer  $l$  and  $v \in \mathcal{I}$ . In particular,  $f_l^{(0)} = Id$ .

With the help of the representations of a quiver and Theorem 3.5, we can describe the category of the comodules of  $U_q(sl_2)$ .

**Theorem 4.3.** Assume that  $q$  is not a root of unity. Then there is an equivalence between the category of the right  $U_q(sl_2)$ -comodules and the full subcategory of  $\text{Rep}(k, \mathcal{D})$  whose objects  $V = (V_l, f_a : e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$  satisfies the following conditions:

- (i)  $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$  for all  $l \in \mathbb{Z}$ .
- (ii) For any  $m \in V_l$ ,  $f_l^{(v)}(m) = 0$  for all but finitely many  $v \in \mathcal{I}$ .

**Proof.** By Theorem 3.5, as a coalgebra  $U_q(sl_2)$  is isomorphic to the subcoalgebra  $\mathcal{C}$  of path coalgebra  $k\mathcal{D}^c$  with the set of basis

$$\left\{ b(l, n, i) := \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.$$

For a coalgebra  $C$ , let  $\mathcal{M}^C$  denote the category of the right  $C$ -comodules. So we have the following embedding of categories

$$\mathcal{M}^{U_q(sl_2)} \simeq \mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \text{Rep}(k, \mathcal{D}),$$

where  $\mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c}$  since  $\mathcal{C}$  is a subcoalgebra of  $k\mathcal{D}^c$ , and  $\mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \text{Rep}(k, \mathcal{D})$  is the embedding described in 4.1.

Now, the question is reduced to determine all locally nilpotent  $k$ -representations of quiver  $\mathbf{D}$  which are right  $\mathcal{C}$ -comodules, via the equivalence described in 4.1.

It follows from the definition that a representation  $V = (V_l, f_a : e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$  of the quiver  $\mathbf{D}$  is locally nilpotent if and only if condition (ii) is satisfied. Assume that such a  $V$  is locally nilpotent, then  $M = \bigoplus_{l \in \mathbb{Z}} V_l$  becomes a right  $k\mathcal{D}^c$ -comodule via

$$\rho(m) = \sum_{v \in \mathcal{I}} f_l^{(v)}(m) \otimes P_l^{(v)} \in M \otimes k\mathcal{D}^c$$

for all  $m \in V_l, l \in \mathbb{Z}$ .

If for an arbitrary fixed  $m \in V_l, l \in \mathbb{Z}$ , the element  $\frac{f_l^{(v)}(m)}{\chi(v)}$  only depends on  $|v|$  and  $|T_v|$ , then we can write

$$\begin{aligned} \rho(m) &= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \sum_{v \in I^n, |T_v|=i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v) P_l^{(v)} \\ &= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \left( \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \right) \\ &\in M \otimes \mathcal{C}, \end{aligned}$$

and hence  $M$  becomes a right  $\mathcal{C}$ -comodule. Conversely, if  $M$  becomes a right  $\mathcal{C}$ -comodule, then we have

$$\begin{aligned} \rho(m) &= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \sum_{v \in I^n, |T_v|=i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v) P_l^{(v)} \\ &= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} m(n, i) \otimes \left( \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \right) \end{aligned}$$

for some  $m(n, i) \in M$ . Since

$$\{\chi(v)P_l^{(v)} \mid v \in I^n, |T_v| = i, 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\}$$

is a basis of  $k\mathcal{D}^c$ , it follows that

$$m(n, i) = \frac{f_l^{(v)}(m)}{\chi(v)},$$

which implies that  $\frac{f_l^{(v)}(m)}{\chi(v)}$  only depends on  $|v|$  and  $|T_v|$  for an arbitrary fixed  $m \in V_l, l \in \mathbb{Z}$ .

Now, condition (i) implies that for an arbitrary fixed  $m \in V_l, l \in \mathbb{Z}$ , the element  $\frac{f_l^{(v)}(m)}{\chi(v)}$  only depends on  $|v|$  and  $|T_v|$ . Conversely, by taking  $v = (-1, 1)$  and  $v' = (1, -1)$  in  $\mathcal{I}$  we obtain

$$\frac{f_l^{(-1,1)}(m)}{\chi((-1, 1))} = \frac{f_l^{(1,-1)}(m)}{\chi((1, -1))},$$

which is exactly condition (i). This completes the proof. ■

**Theorem 4.3** permits us to explicitly construct some  $U_q(sl_2)$ -comodules. In the following  $q$  is not a root of unity.

**Example 4.4.** Let  $A$  be the quantum plane generated by  $X$  and  $Y$  subject to the relation  $XY = q^2 YX$ . Let  $l$  be an integer and  $n$  a non-negative integer. Then for any  $A$ -module  $U$  one can define a representation  $V = V_{(l,n,U)}$  of quiver **D** as follows:

$$\begin{aligned} V_j &:= U, & \text{if } l \leq j \leq l+n, \\ V_j &:= 0, & \text{otherwise;} \\ f_j^{(1)} &:= X, & \text{if } l+1 \leq j \leq l+n, \\ f_j^{(1)} &:= 0, & \text{otherwise;} \\ f_j^{(-1)} &:= Y, & \text{if } l+1 \leq j \leq l+n, \\ f_j^{(-1)} &:= 0 & \text{otherwise} \end{aligned}$$

where  $l$  is any integer and  $n \geq 0$ . Then by **Theorem 4.3**,  $V$  induces a right  $U_q(sl_2)$ -comodule.

**Example 4.5.** Let  $l$  be an integer and  $n$  a non-negative integer.

(i) For each  $\lambda \in k$ , one can define a representation  $V$  of quiver **D** as follows:

$$\begin{aligned} V_j &:= k, & \text{if } l \leq j \leq l+n, \\ V_j &:= 0, & \text{otherwise;} \\ f_j^{(1)} &:= 1, & \text{if } l+1 \leq j \leq l+n, \\ f_j^{(1)} &:= 0, & \text{otherwise;} \\ f_j^{(-1)} &:= \lambda q^{-2(l+n-j)}, & \text{if } l+1 \leq j \leq l+n, \\ f_j^{(-1)} &:= 0, & \text{otherwise.} \end{aligned}$$

Then by **Theorem 4.3**,  $V$  induces a right  $U_q(sl_2)$ -comodule, which is denoted by  $M_{(l,n,\lambda)}$ .

(ii) Consider the representation  $V$  of quiver **D** defined by:

$$\begin{aligned} V_j &:= k, & \text{if } l \leq j \leq l+n, \\ V_j &:= 0, & \text{otherwise;} \\ f_j^{(1)} &:= 0, & \forall j \in \mathbb{Z}; \\ f_j^{(-1)} &:= 1, & \forall j \in \mathbb{Z}. \end{aligned}$$

Then by **Theorem 4.3**,  $V$  induces a right  $U_q(sl_2)$ -comodule, which is denoted by  $M_{(l,n,\infty)}$ .

## 4.6

A finite-dimensional right  $U_q(sl_2)$ -comodule  $(M, \rho)$  is said to be Schurian, if  $\dim_k M_j = 1$  or  $0$  for each integer  $j$ , where  $M_j := \{m \in M \mid (Id \otimes \pi_0)\rho(m) = m \otimes e_j\}$  and  $\pi_0$  is the projection from  $k\mathcal{D}^c$  to  $k\mathcal{D}_0$ .

**Theorem 4.7.** When the triple  $(l, n, \lambda)$  runs over  $\mathbb{Z} \times \mathbb{N}_0 \times (k \cup \{\infty\})$ ,  $M_{(l,n,\lambda)}$  gives a complete list of all pairwise non-isomorphic, indecomposable Schurian right  $U_q(sl_2)$ -comodules, where  $q$  is not a root of unity.

**Proof.** Assume that  $M$  is an indecomposable Schurian right  $U_q(sl_2)$ -comodule. Set  $\text{Supp}(M) := \{j \in \mathbb{Z} \mid M_j \neq 0\}$ . Let  $l$  and  $l+n$  be the minimal and the maximal elements in  $\text{Supp}(M)$ . Then  $\text{Supp}(M) \subseteq \{l, l+1, \dots, l+n\}$ . We claim that  $\text{Supp}(M) = \{l, l+1, \dots, l+n\}$ .

Otherwise, there exists a  $j_0$  such that  $l < j_0 < l+n$  and  $j_0 \notin \text{Supp}(M)$ . Then by (4.1) we have a  $k\mathcal{D}_0$ -comodule decomposition

$$M = \left( \bigoplus_{j < j_0} M_j \right) \oplus \left( \bigoplus_{j > j_0} M_j \right).$$

Since  $M_{j_0} = 0$ , it follows that this is a  $k\mathcal{D}^c$ -comodule decomposition, and hence it is also a  $U_q(sl_2)$ -comodule decomposition, which contradicts the assumption.

Note that each  $M_j$  is one dimensional for  $l \leq j \leq l+n$ . Set

$$a_j := f_j^{(1)} \quad \text{and} \quad b_j := f_j^{(-1)}, \quad l+1 \leq j \leq l+n.$$

Note that for each  $j$ , we have  $a_j \neq 0$  or  $b_j \neq 0$  (otherwise, say  $a_{j_0} = b_{j_0} = 0$ , then we again have a  $U_q(sl_2)$ -comodule decomposition  $M = (\bigoplus_{j < j_0} M_j) \oplus (\bigoplus_{j \geq j_0} M_j)$ ).

By Theorem 4.3 we have  $a_j b_{j+1} = q^2 b_j a_{j+1}$  for all  $j$  with  $l+1 \leq j \leq l+n-1$ . Now, if some  $b_{j_0} = 0$ , then all  $b_j = 0$  and all  $a_j \neq 0$ , and hence  $M$  is isomorphic to  $M_{(l,n,0)}$ . If some  $a_{j_0} = 0$ , then all  $a_j = 0$  and all  $b_j \neq 0$ , and hence  $M$  is isomorphic to  $M_{(l,n,\infty)}$ . If  $a_j \neq 0 \neq b_j$  for all  $j$ , then  $M$  is isomorphic to  $M_{(l,n,\lambda)}$  with  $\lambda = \frac{b_{l+1}}{a_{l+1}} q^{2(n-1)}$ .

On the other hand, each  $M_{(l,n,\lambda)}$  is indecomposable since its socle is of dimension one, and they are clearly pairwise non-isomorphic. ■

## 5. A class of $SL_q(2)$ -modules

Theorem 4.3 characterizes the category of the right  $U_q(sl_2)$ -comodules by a full subcategory of the category of the  $k$ -representations of  $\mathbf{D}$ , where  $q$  is not a root of unity, and  $\mathbf{D}$  is the Gabriel quiver of  $U_q(sl_2)$  as a coalgebra. This permits us to construct some left  $SL_q(2)$ -modules from some representations of quiver  $\mathbf{D}$ , via the duality between  $U_q(sl_2)$  and  $SL_q(2)$ .

## 5.1

Recall that the algebra homomorphism  $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$  in Lemma 2.3 is given by

$$\begin{aligned} \psi(a)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,0} q^l + \delta_{i,1} \delta_{j,1} q^{l-1}, & \psi(b)(K^l E^i F^j) &= \delta_{i,1} \delta_{j,0} q^{l-1}, \\ \psi(c)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,1} q^{-l}, & \psi(d)(K^l E^i F^j) &= \delta_{i,0} \delta_{j,0} q^{-l}, \end{aligned}$$

where  $E' = K^{-1}E$ .

Let  $(M, \rho)$  be a right  $U_q(sl_2)$ -comodule. Then  $M$  becomes a left  $SL_q(2)$ -module via

$$x.m := \sum \psi(x)(m_1)m_0, \tag{5.1}$$

for  $x \in SL_q(2)$ ,  $m \in M$ , where  $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes U_q(sl_2)$ .

Let  $\mathcal{C}$  be the subcoalgebra of  $k\mathcal{D}^c$  with the set of basis

$$\left\{ b(l, n, i) = \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)} \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.$$

Identifying  $U_q(sl_2)$  with  $\mathcal{C}$  via (3.1), we can evaluate  $\psi(a)$ ,  $\psi(b)$ ,  $\psi(c)$  and  $\psi(d)$  on this set of basis of  $\mathcal{C}$  via Lemma 2.3. Since

$$b(l, n, i) = \frac{q^{i(i+1)}}{i!_q j!_q} \theta(K^l E'^i F^j) \quad \text{with } j = n - i,$$

it follows that the list of the non-zero values is as follows:

$$\begin{aligned} \psi(a)(b(l, 0, 0)) &= \psi(a)(K^l) = q^l, \\ \psi(a)(b(l, 2, 1)) &= \psi(a)(q^2 K^l E' F) = q^{l+1}, \\ \psi(b)(b(l, 1, 1)) &= \psi(b)(q^2 K^l E') = q^{l+1}, \\ \psi(c)(b(l, 1, 0)) &= \psi(c)(K^l F) = q^{-l}, \\ \psi(d)(b(l, 0, 0)) &= \psi(d)(K^l) = q^{-l}. \end{aligned}$$

**Theorem 5.2.** Let  $V = (V_l, f_a : l \in \mathcal{D}_0, a \in \mathcal{D}_1)$  be a  $k$ -representation of the quiver  $\mathbf{D}$  satisfying the following conditions:

- (i)  $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$  for all  $l \in \mathbb{Z}$ .
- (ii) For any  $m \in V_l$ ,  $f_l^{(v)}(m) = 0$  for all but finitely many  $v \in \mathcal{I}$ , where  $f_l^{(v)} = f_{P_l^{(v)}}$ ,  $f_l^{(0)} = Id|_{V_l}$ .

Then  $M = \bigoplus_{l \in \mathbb{Z}} V_l$  is a left  $SL_q(2)$ -module via

$$\begin{aligned} a.m &= q^l m + q^{l-1} f_l^{(1, -1)}(m), \\ b.m &= q^{l-1} f_l^{(1)}(m), \\ c.m &= q^{-l} f_l^{(-1)}(m), \\ d.m &:= q^{-l} m, \end{aligned}$$

for each  $m \in V_l$ ,  $l \in \mathbb{Z}$ , where  $q$  is not a root of unity.

**Proof.** By Theorem 4.3  $M = \bigoplus_{l \in \mathbb{Z}} V_l$  is a right  $U_q(sl_2)$ -comodule via

$$\rho(m) = \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes b(l, n, i)$$

where  $v$  is a fixed element in  $I^n$  with  $|T_v| = i$ , and  $m \in V_l$ ,  $l \in \mathbb{Z}$ . By (5.1),  $M$  becomes a left  $SL_q(2)$ -module via

$$x.m := \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \psi(x)(b(l, n, i)) \frac{f_l^{(v)}(m)}{\chi(v)}.$$

It follows that for each  $m \in V_l$ ,  $l \in \mathbb{Z}$ , we have

$$\begin{aligned} a.m &= \psi(a)(b(l, 0, 0))m + \psi(a)(b(l, 2, 1)) \frac{f_l^{(1, -1)}(m)}{\chi(1, -1)} \\ &= q^l m + q^{l-1} f_l^{(1, -1)}(m), \\ b.m &= \psi(b)(b(l, 1, 1)) \frac{f_l^{(1)}(m)}{\chi(1)} = q^{l-1} f_l^{(1)}(m), \\ c.m &= \psi(c)(b(l, 1, 0)) \frac{f_l^{(-1)}(m)}{\chi(0)} = q^{-l} f_l^{(-1)}(m), \\ d.m &:= \psi(d)(b(l, 0, 0))m = q^{-l} m. \quad \blacksquare \end{aligned}$$

Theorem 5.2 permits us to write out explicitly the following examples of  $SL_q(2)$ -modules.

**Example 5.3.** Let  $A$  be the quantum plane generated by  $X$  and  $Y$  subject to the relation  $XY = q^2YX$ , and  $U$  be a left  $A$ -module, where  $q$  is not a root of unity. Let  $l$  be an integer and  $n$  a non-negative integer. For any element  $u \in U$  and  $1 \leq i \leq n+1$ , let  $U^{n+1}$  denote the direct sum of the copies of  $U$ , and  $u_i$  denote the element in  $U^{n+1}$  with the  $i$ th component being  $u$  and other components being zero. Then by Theorem 5.2 and Example 4.4, the copy  $U^{n+1}$  becomes a left  $SL_q(2)$ -module with the following actions:

$$\begin{aligned} au_i &= q^{i+l-1}u_i + q^{i+l-2}YXu_{i-2}, \quad 3 \leq i \leq n+1, \quad au_i = 0, \text{ otherwise,} \\ bu_i &= q^{i+l-2}Xu_{i-1}, \quad 2 \leq i \leq n+1, \quad bu_i = 0, \text{ otherwise,} \\ cu_i &= q^{-(i+l-1)}Yu_{i-1}, \quad 2 \leq i \leq n+1, \quad cu_i = 0, \text{ otherwise,} \\ du_i &= q^{-(i+l-1)}u_i, \quad \forall i. \end{aligned}$$

**Example 5.4.** Let  $V$  be a  $k$ -space of dimension  $n+1$ ,  $n \in \mathbb{N}_0$ , with basis  $v_0, v_1, \dots, v_n$ . Let  $l$  be an integer, and  $q \in k$  be not a root of unity.

(i) Let  $\lambda \in k$ . Then by Theorem 5.2 and Example 4.5(i),  $V$  becomes a left  $SL_q(2)$ -module via the following actions, which is denoted again by  $M_{(l,n,\lambda)}$

$$\begin{aligned} a.v_i &= q^{l+i}v_i + \lambda q^{-2n+l+3i-3}v_{i-2}, \quad 2 \leq i \leq n, \\ a.v_i &= q^{l+i}v_i, \quad i = 0, 1, \\ b.v_i &= q^{l+i-1}v_{i-1}, \quad 1 \leq i \leq n, \\ b.v_0 &= 0, \\ c.v_i &= \lambda q^{-2n+i-l}v_{i-1}, \quad 1 \leq i \leq n, \\ c.v_0 &= 0, \\ d.v_i &= q^{-(l+i)}v_i, \quad \forall i. \end{aligned}$$

(ii) By Theorem 5.2 and Example 4.5(ii),  $V$  also becomes a left  $SL_q(2)$ -module via the following actions, which is denoted again by  $M_{(l,n,\infty)}$

$$\begin{aligned} a.v_i &= q^{l+i}v_i, \quad \forall i, \\ b.v_i &= 0, \quad \forall i, \\ c.v_i &= q^{-(l+i)}v_{i-1}, \quad 1 \leq i \leq n, \\ c.v_0 &= 0, \\ d.v_i &= q^{-(l+i)}v_i, \quad \forall i. \end{aligned}$$

Note that  $M_{(l,n,\lambda)}$  with  $l \in \mathbb{Z}$ ,  $n \in \mathbb{N}_0$ ,  $\lambda \in k \cup \{\infty\}$  are indecomposable, pairwise non-isomorphic  $SL_q(2)$ -modules.

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